Interpreting Prediction Market Prices as Probabilities

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Research Question

- Our question: What is the relationship between prediction markets prices and traders' beliefs?
- Some motivation:
  - "What is the logical basis for interpreting the price of an all-or-nothing futures contract as a market probability that the event will occur?"
    - Charles Manski
      "Interpreting the Predictions of Prediction Markets", NBER WP #10359
- Further observations:
  - "Researchers engaged in empirical study of prediction markets have been uncomfortably vague."
  - "Recent papers on prediction markets provide no formal analysis showing how such markets aggregate information or opinions"

Model Ingredients

- Commodity: Contract pays $1 if event occurs
- Many traders, each characterized by:
  - q: *S*ubjective beliefs
  - y: Wealth
  - U: Utility function
  - $\Rightarrow$Implies: Individual demand: $x(\pi; q, y, U)$
- Aggregates to demand/supply functions:
  - π: Price of contract
  - Aggregate Demand and Supply of contracts
- Equilibrium condition: $\sum x(\pi)=0$

Model Ingredients: Manski

- Commodity: Contract pays $1 if event occurs
- Many traders, each characterized by:
  - *q: S*ubjective beliefs — General distribution: ~F(q,y)
  - *y:* Wealth — Orthogonal to beliefs: E[q,y]=0
  - *U:* Utility function — Always bet exactly $y

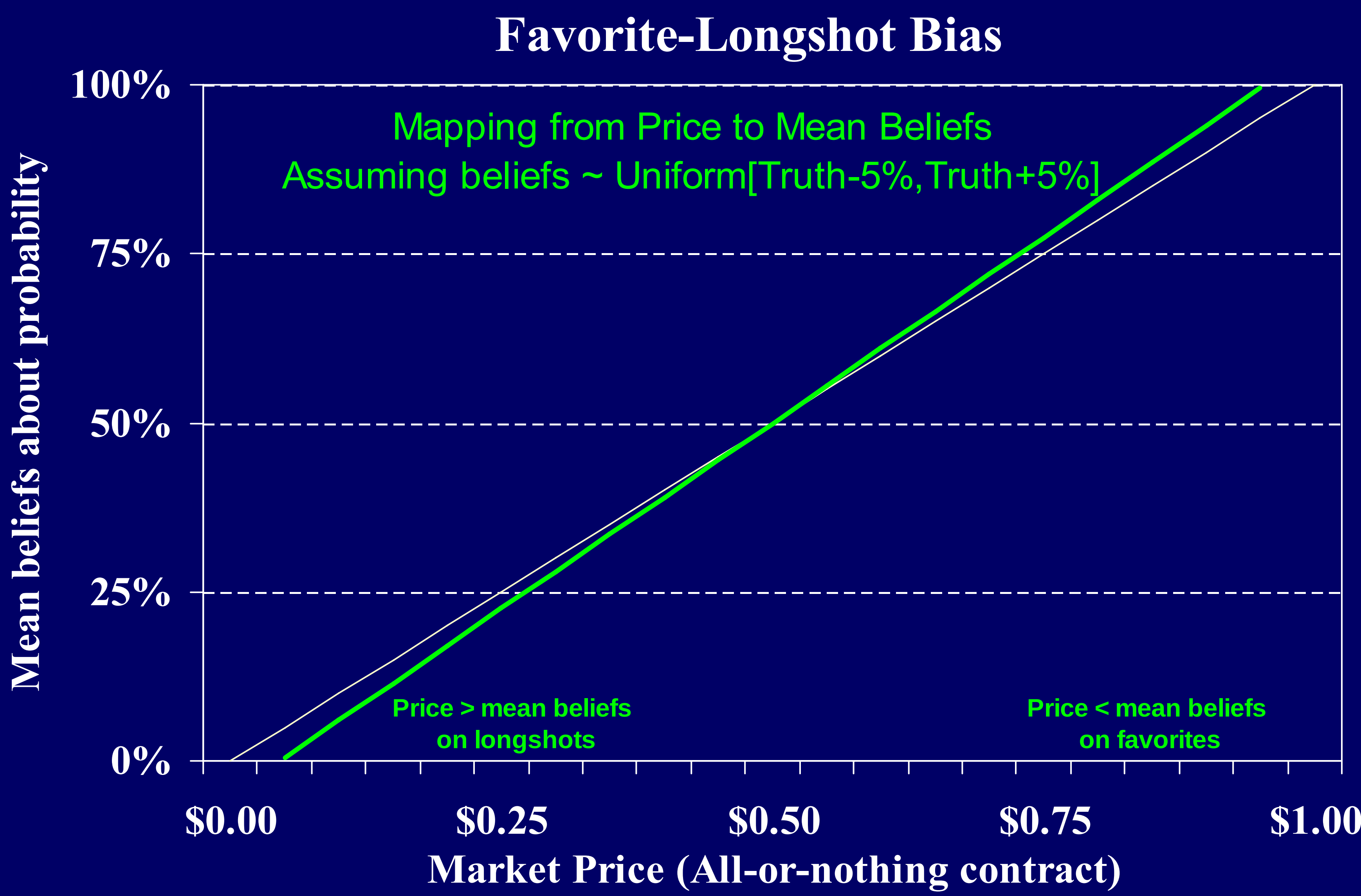
  ⇒Implies: Individual demand: *x(*π; *q, y, U)*

$$x = \begin{cases} y/\pi & \text{if } q>\pi \\ -y/(1-\pi) & \text{if } q<\pi \end{cases}$$

- Aggregates to demand/supply functions:
  - π: Price of contract
  - Aggregate Demand and Supply of contracts
    $1[q>\pi]\ (y/\pi) = 1[q<\pi]\ (y/(1-\pi))$
- Equilibrium condition: $\sum x(\pi)=0$
  - $[1-F(\pi)]/\pi = F(\pi)/(1-\pi)$

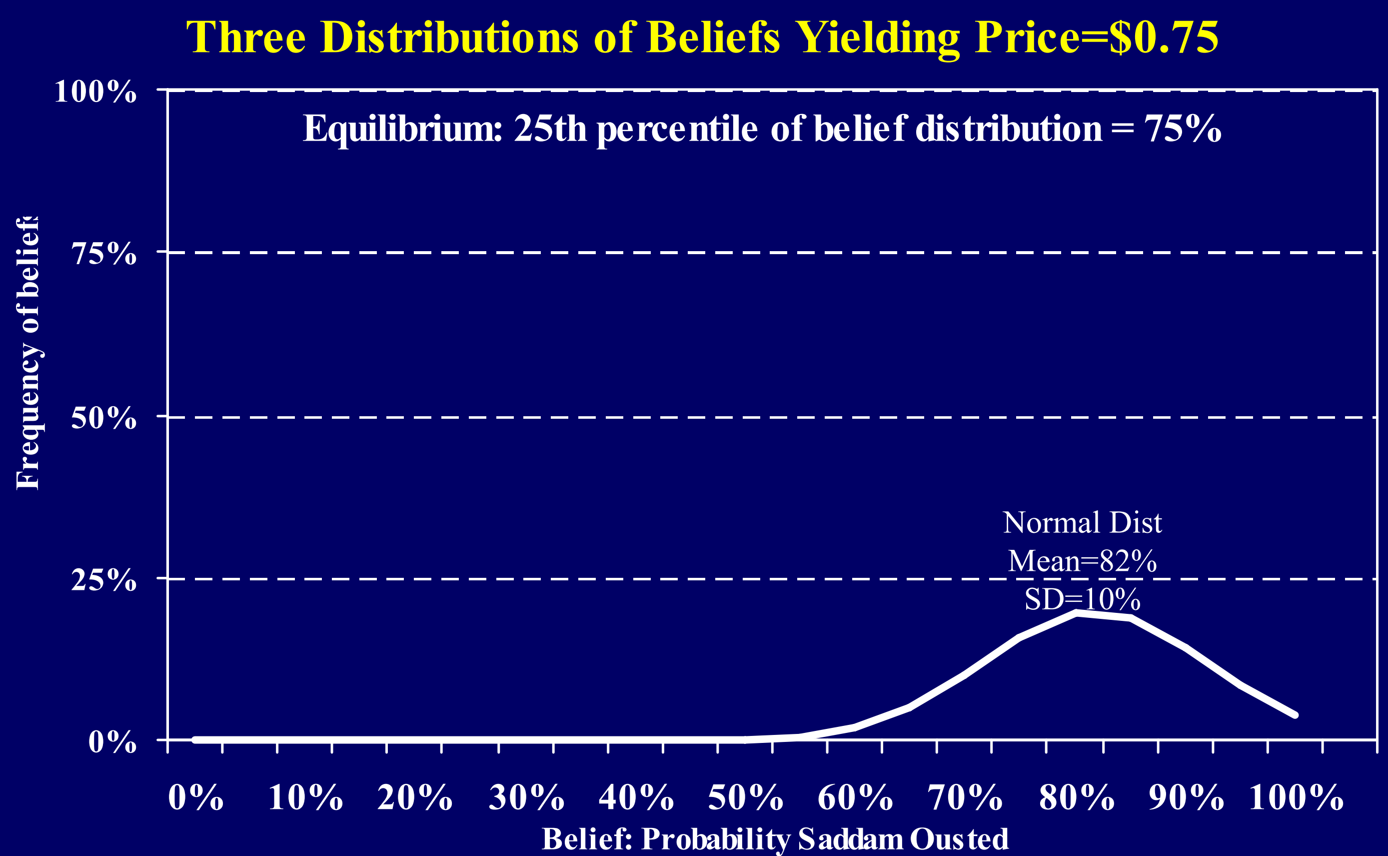
    $\Rightarrow \pi = F(1-\pi)$

What do Prices Say About Beliefs?



Manski Model: Implications

- Example: Saddam Security Price=$0.75
  ⇒25% of traders believe probability<75%



- Note:
  - "Price does not generally equal the mean belief of traders"
  - "π is the midpoint of an interval of width $2(\pi - \pi^2)$ that contains $E(q)$"

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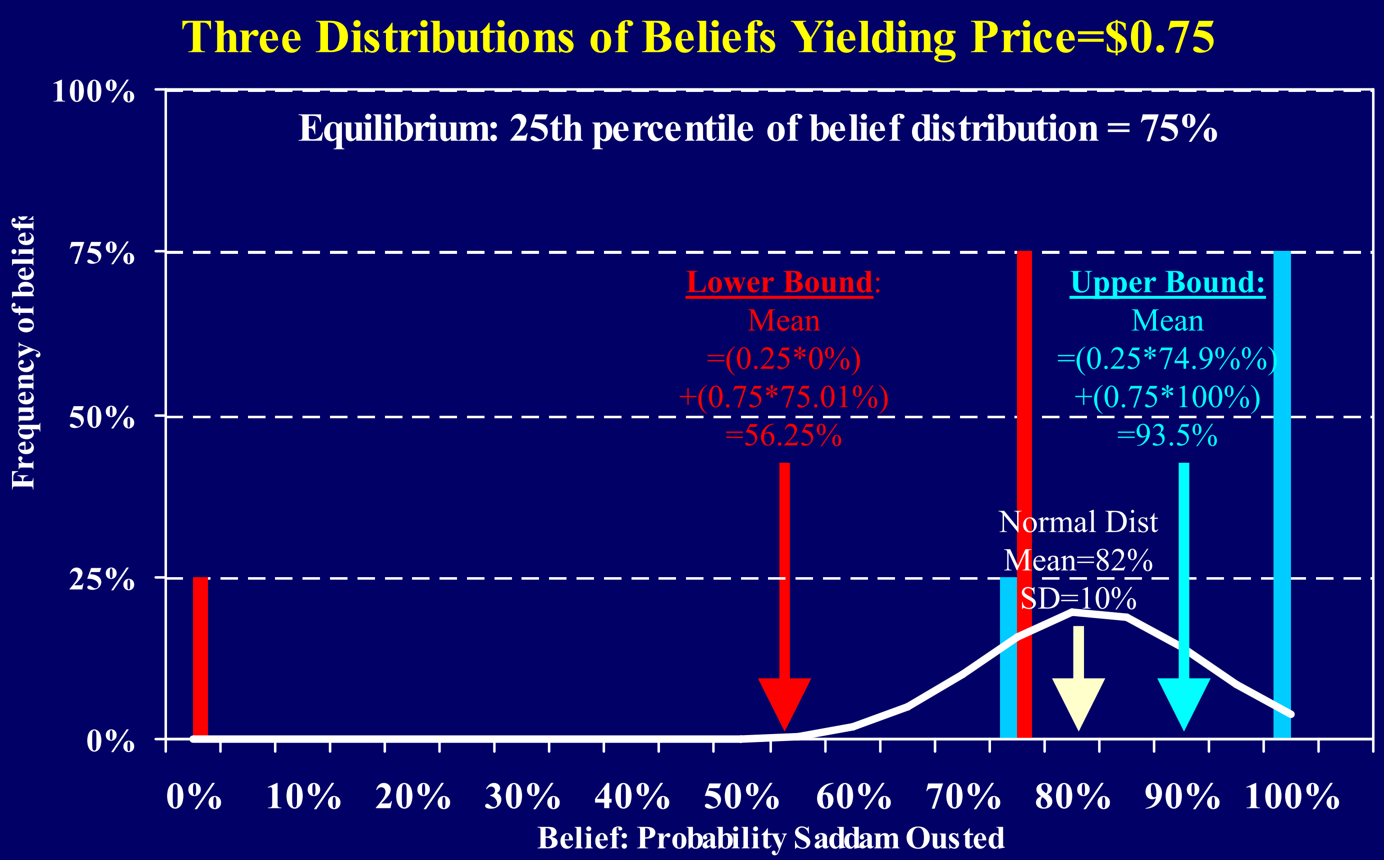
- Example: Saddam Security Price=$0.75
  ⇒75% of traders believe probability>75%



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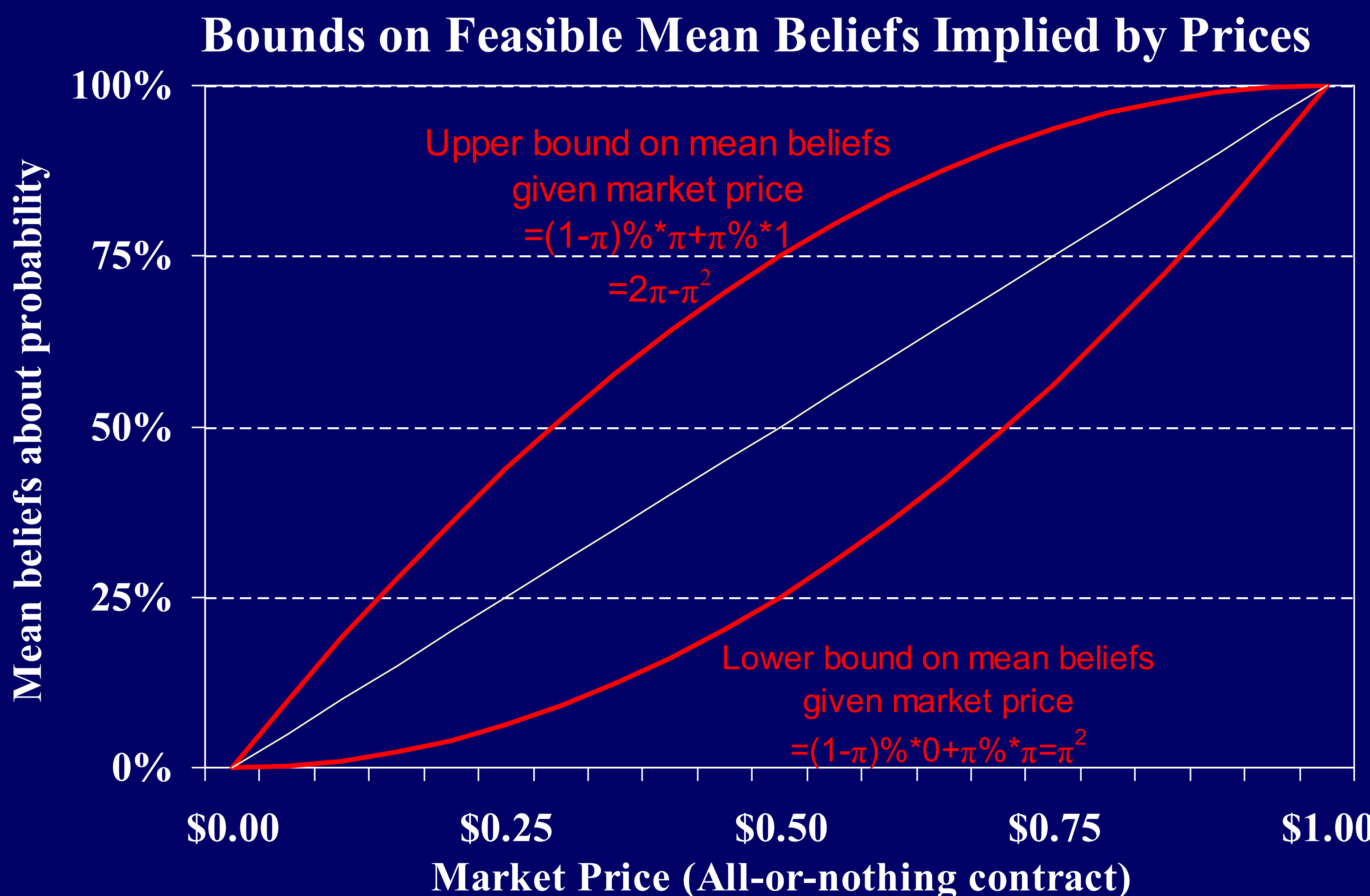


Implications (According to Manski)

- "Refutes the notion that prices in prediction markets are 'market probabilities'"
- "The price of an all-or-nothing futures contract does not equal the mean, median, or any other measure of the central tendency of traders' beliefs"
- "Prices near zero or one are very informative about the mean beliefs of traders, but prices near 0.5 are much less informative"

Model Ingredients: This Paper

- Commodity: Contract pays $1 if event occurs
- Many traders, each characterized by:
  - *q: S*ubjective beliefs — General distribution: ~F(q,y)
  - *y:* Wealth — Consider E[*yq*]≠0
  - *U:* Utility function — Consider many utility functions
  - ⇒Implies: Individual demand: $x(\pi; q, y, U)$ — Endogenize choice of bet size
- Aggregates to demand/supply functions:
  - π: Price of contract
  - Aggregate Demand and Supply of contracts
- Equilibrium condition: $\sum x(\pi)=0$

Simple Model: Log Utility

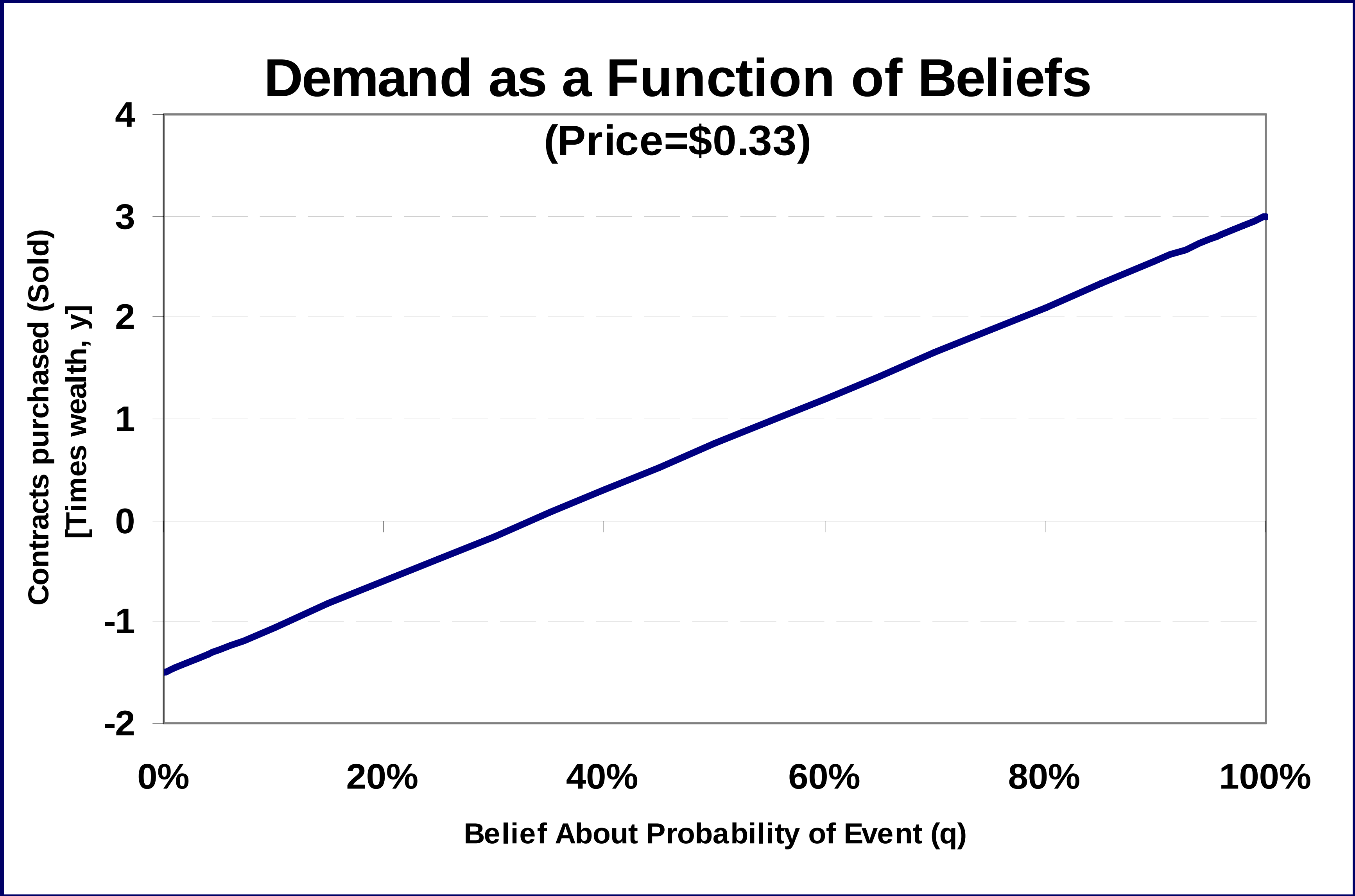
- Individual Demand:

$$\underset{\{x\}}{Max}\ EU_j = q_j Log(y + x_j(1-\pi)) + (1-q_j) Log(y - x_j \pi)$$

$$yielding: \quad x_j^* = y \frac{q_j - \pi}{\pi(1-\pi)}$$

- Individual demand is:
  - *Symmetric* in beliefs relative to price, *q-π*
  - *Linear* in beliefs, *q*
  - *Decreasing in risk, π(1-π)*
  - *Homothetic* in income, *y*
  - *Unique* for prices between 0 and 1

Individual Demand (Log Utility)



Equilibrium

- Supply = Demand (Assuming y⊥q)

$$\int_{-\infty}^{\pi} y \frac{q-\pi}{\pi(1-\pi)} f(q)dq = \int_{\pi}^{\infty} y \frac{\pi-q}{\pi(1-\pi)} f(q)dq$$

- Implies: *Price = Mean belief*

$$\pi = \int_{-\infty}^{\infty} q f(q)dq = \overline{q}$$

- And if beliefs (q) are correlated with wealth (y)

$$\int y \frac{q-\pi}{\pi(1-\pi)} dF(q \le \pi, y) = \int y \frac{\pi-q}{\pi(1-\pi)} dF(q \ge \pi, y)$$

$$\pi = \int q \frac{y}{\overline{y}} dF(q, y)$$

=*Wealth-weighted mean belief*

General Set-Up

$$\underset{\{x\}}{Max}\ EU_j = q_j U(y + x_j(1-\pi)) + (1-q_j)U(y - x_j\pi)$$

$$FOC: \quad \frac{U'(y + x - \pi x)}{U'(y - \pi x)} = \frac{\pi}{1-\pi}\frac{(1\text{-}q)}{q}$$

$$Implies{:}\ x^* = X(q,\pi,y)\ and\ sign(\frac{\partial X}{\partial \pi}) = -sign(\frac{\partial X}{\partial q})$$

Specific Utility Functions

Table 1: Utility Functions and Demand for Prediction Securities

Utility Function	Utility	Demand
Log utility (CRRA with γ = 1)	$u(y)=\ln(y)$	$\frac{y}{\pi(1-\pi)}(q-\pi)$
Constant relative risk aversion (CRRA) (γ > 0, γ ≠ 1)	$u(y)=\frac{y^{1-\gamma}}{1-\gamma}$	$\frac{y}{\pi}\cdot\left(\frac{\pi\{[\frac{q(1-\pi)}{\pi(1-q)}]^{\frac{1}{\gamma}}-1\}}{1+\pi\{[\frac{q(1-\pi)}{\pi(1-q)}]^{\frac{1}{\gamma}}-1\}}\right)$
Constant absolute risk aversion (CARA)	$u(y)=-e^{-ry}$	$r^{-1}\cdot[\ln(\frac{q}{1-q})-\ln(\frac{\pi}{1-\pi})]$
Quadratic utility	$u(y)=-\frac{1}{2}(y^{\max}-y)^2$	$(y^{\max}-y)\frac{q-\pi}{q(1-\pi)-\pi(q-\pi)}$
Hyperbolic absolute risk aversion (HARA)*	$u(y)=\frac{1-\gamma}{\gamma}(\frac{ay}{1-\gamma}+b)^{\gamma}$	$\frac{y+b(1-\gamma)a^{-1}}{\pi}\cdot\left(\frac{\pi\{[\frac{\pi(1-q)}{q(1-\pi)}]^{\frac{1}{\gamma-1}}-1\}}{1+\pi\{[\frac{\pi(1-q)}{q(1-\pi)}]^{\frac{1}{\gamma-1}}-1\}}\right)$

* The HARA utility function nests the others as special cases. (For log utility γ→0; risk neutral: γ→1; quadratic: γ = 2; CRRA: γ < 1 and b=0; CARA: γ → -∞ and b > 0).

Are Prices ≈ Mean Beliefs?

Evidence:

1. Calibrate the relationship between prices and probabilities based on *plausible* distributions of beliefs
2. Calibrate the relationship between prices and probabilities for different utility functions based on *known* distributions of beliefs:
   - » Election 2004
   - » NFL Football
3. Explore relationship between prices and endogenous bet sizes
   - » Microdata on bet sizes
4. Analyze relationship between prices and average outcomes
   - » Sports: Baseball, horse racing
   - » Politics
   - » Economic Derivatives

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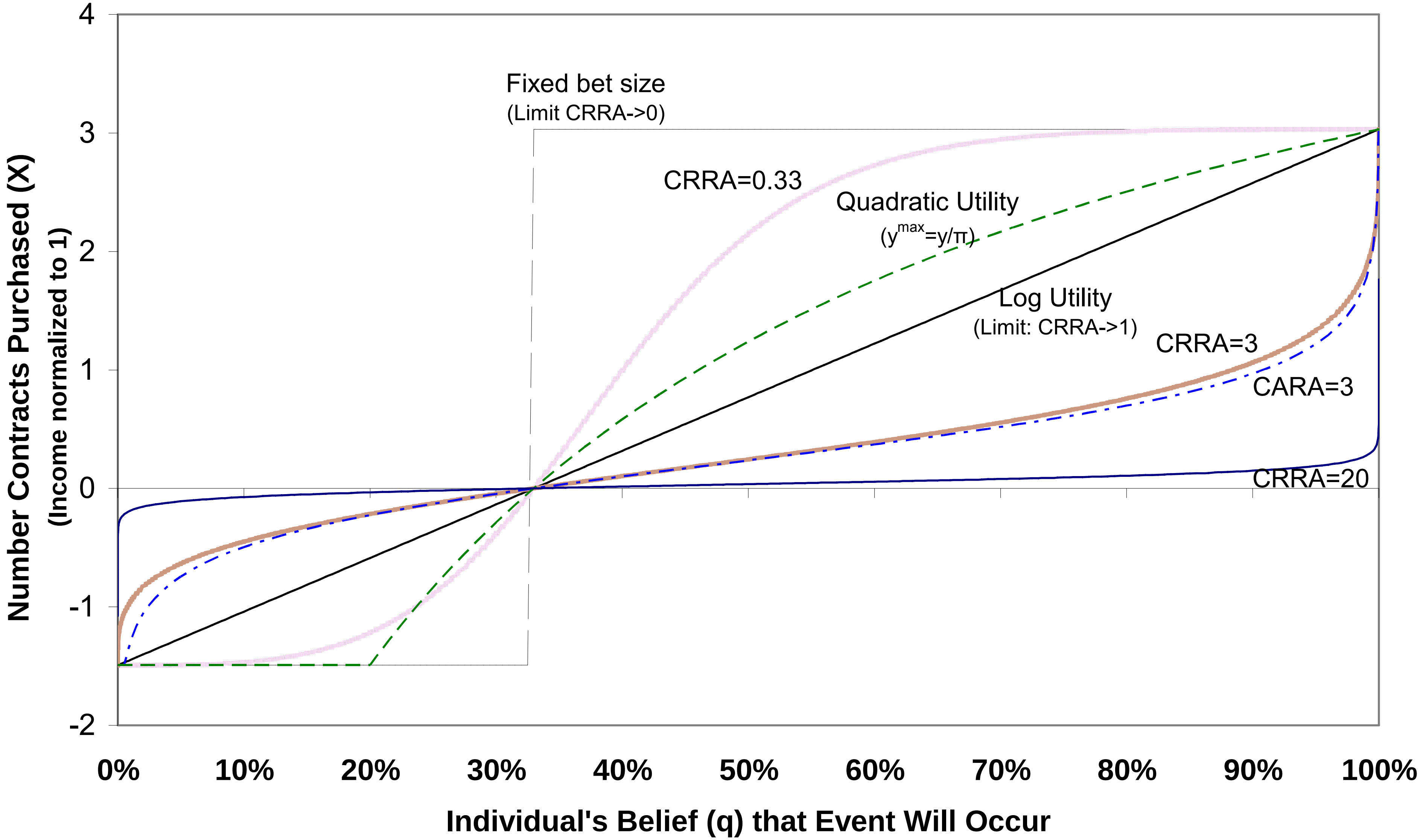
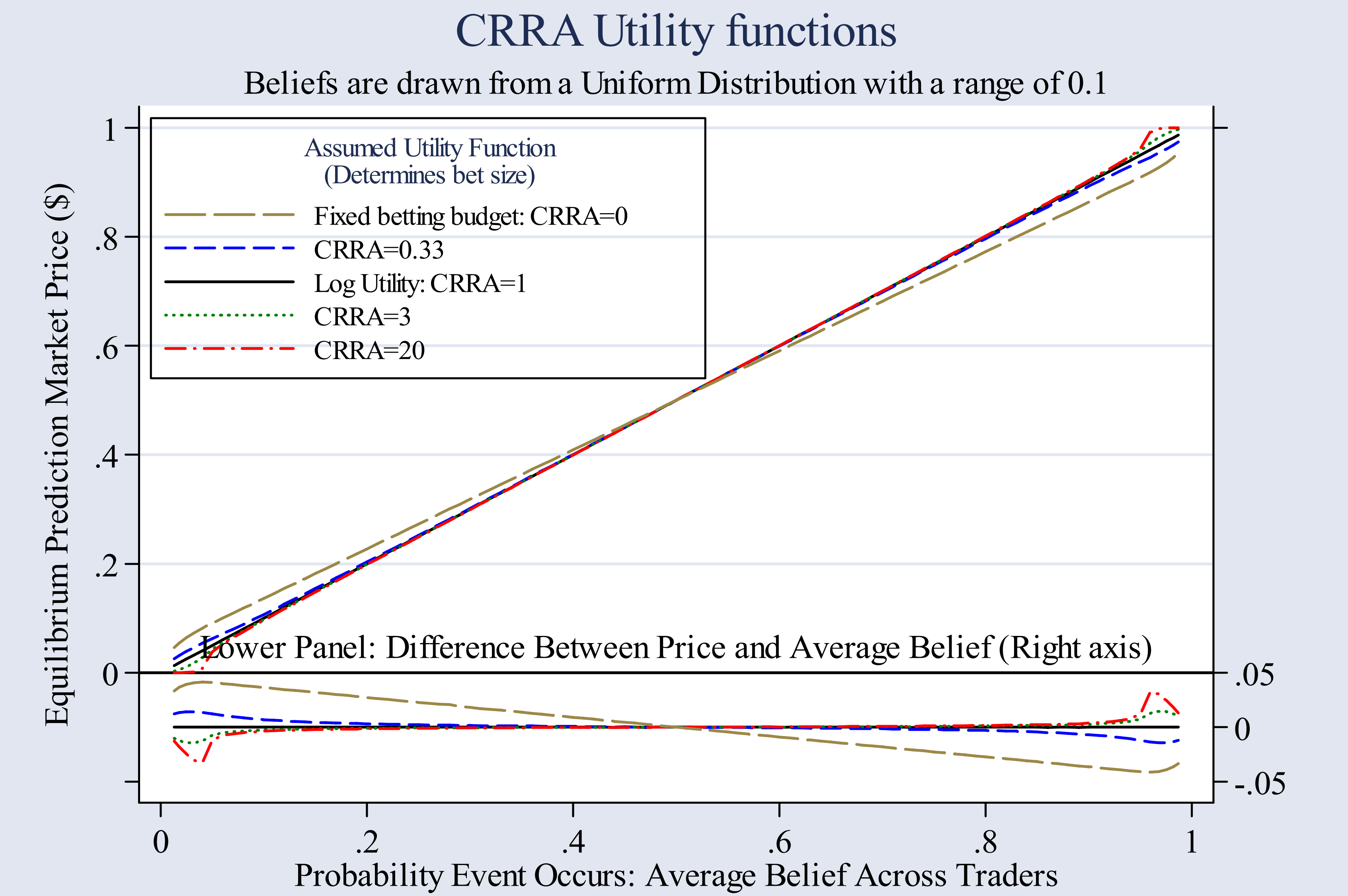
Demand as a Function of Beliefs

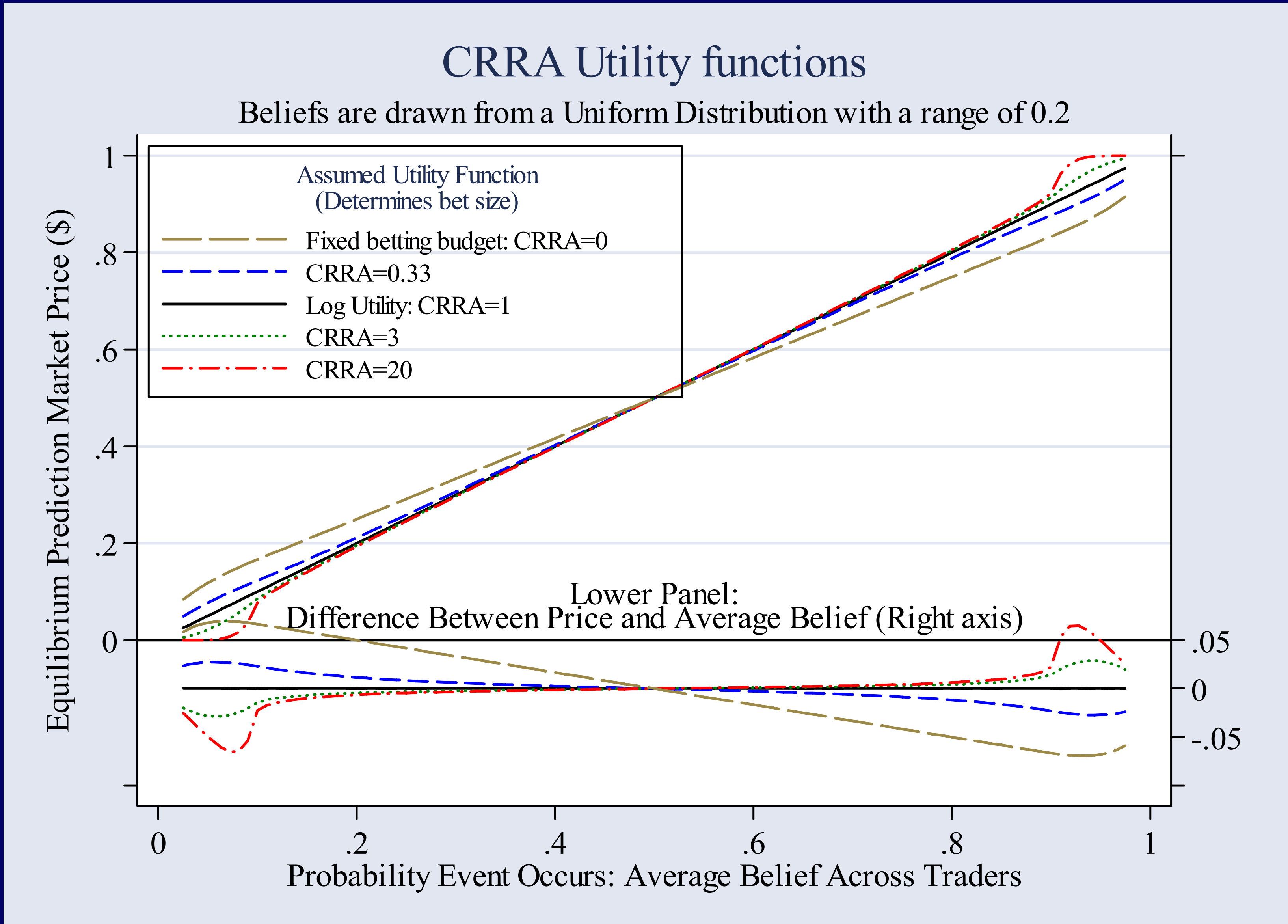
Price=$0.33; Individual demand depends on beliefs and utility function



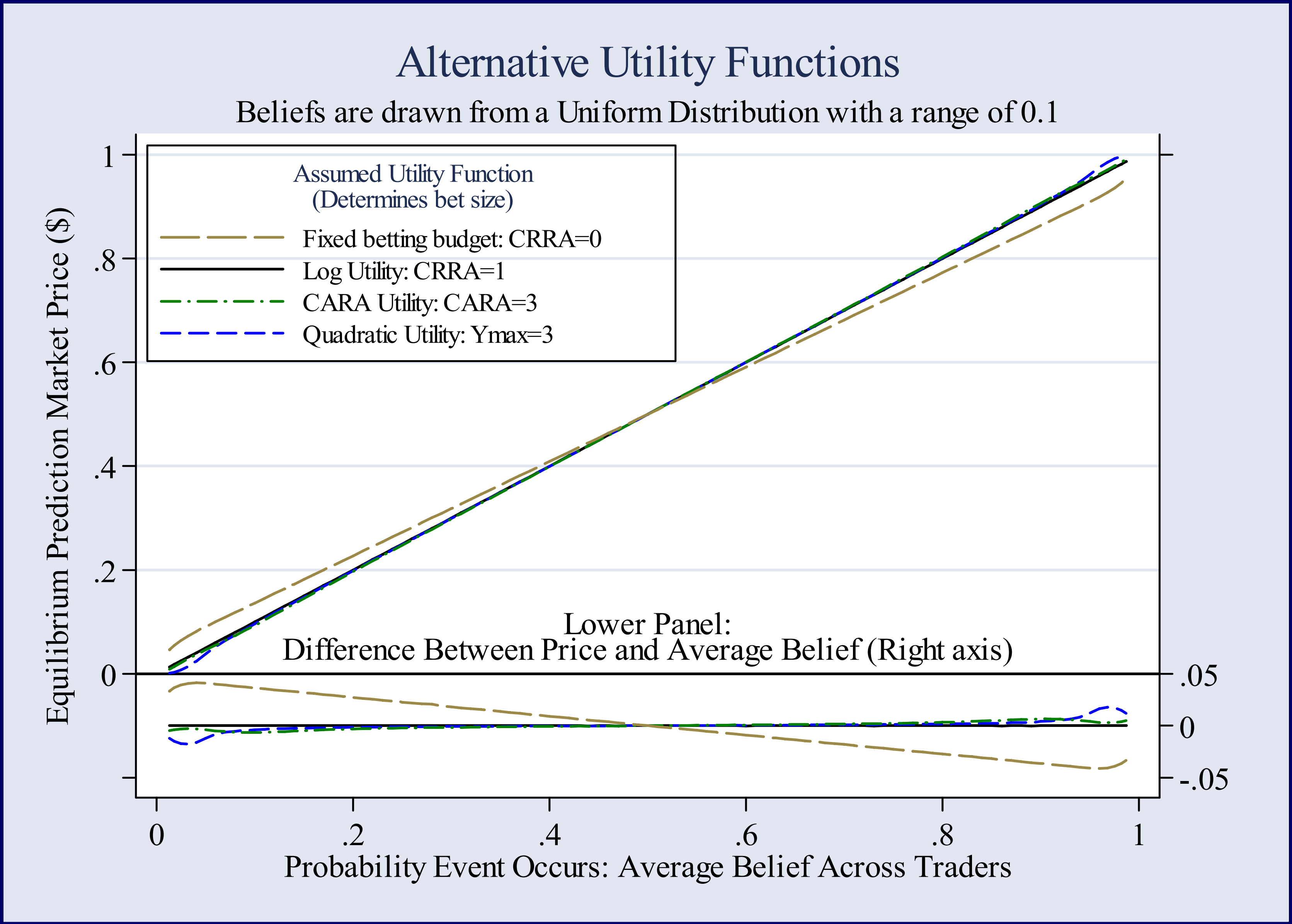
Solving for Equilibrium Prices



Increasing Dispersion of Beliefs



Robustness: Alternative Utility Functions



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Are Prediction Market Prices Close to Probs?

- Election 2004: Two Facts
  1. 62% of Americans thought Bush would win

$$\int_{0.5}^{1} f(q)dq = 0.62$$

  2. Tradesports contract was trading at \$0.55

$$\int_{0}^{0.55} x(q)dF_{\mu,\sigma}(q) = -\int_{0.55}^{1} x(q)dF_{\mu,\sigma}(q)$$

- For plausible distributions of beliefs (*F)* and utility functions *(U*), how close is the market price to the mean belief?

Mean Beliefs Implied by Price=$0.55

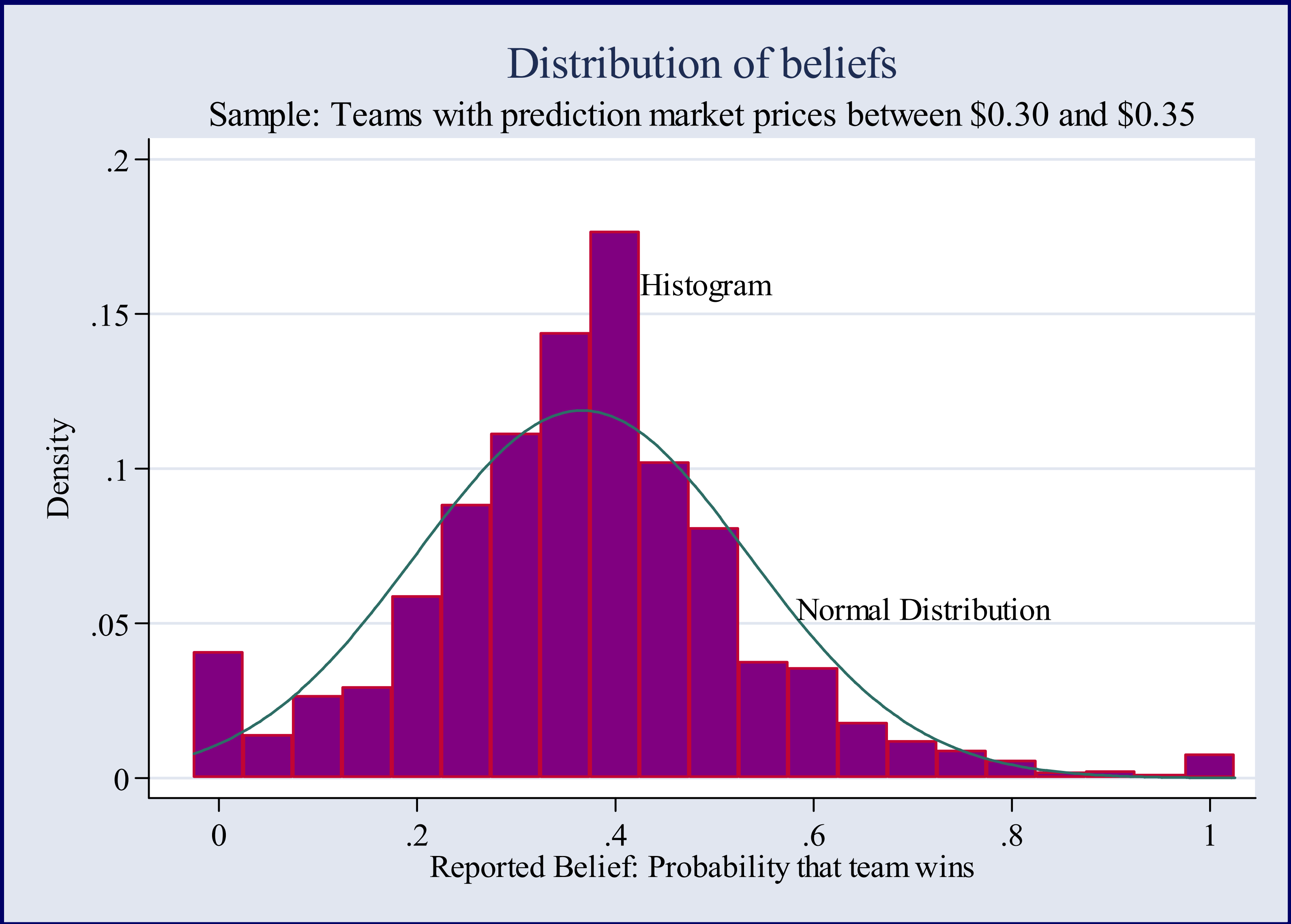
	Normal [μ,σ]	Beta (α,β)	Uniform (q_L,q_H)
Implied Distribution of Beliefs			
Fixed bet size (Limit; γ→0)	57.8% [0.584,0.278]	57.1% [2.112,1.589]	58.6% [0.229,0.942]
CRRA; γ=⅓	56.0% [0.561,0.201]	55.8% [3.370,2.675]	57.5% [0.252,0.897]
Log Utility (γ=1)	55.0% [0.550, 0.163]	55.0% [4.640, 3.804]	55.0% [0.342,0.758]
CRRA; γ=3	54.6% [0.546,0.149]	54.7% [5.337,4.432]	54.9% [0.343,0.755]
CRRA; γ=20	54.4% [0.544,0.144]	54.6% [5.640,4.707]	54.8% [0.345,0.752]
CARA; ρ=3	54.4% [0.544,.0144]	54.5% [5.692,4.754]	54.7% [0.351,0.743]
Quadratic; y^{max}=3	54.2% [0.542,0.138]	54.2% [6.568,5.553]	54.6% [0.351,0.742]

Notes: Table shows mean of distribution. [Parameters of the belief distribution shown in parentheses]
Source: Authors' calculations. Note that beliefs outside (0,1) were treated as lim. q→0 or 1, respectively.

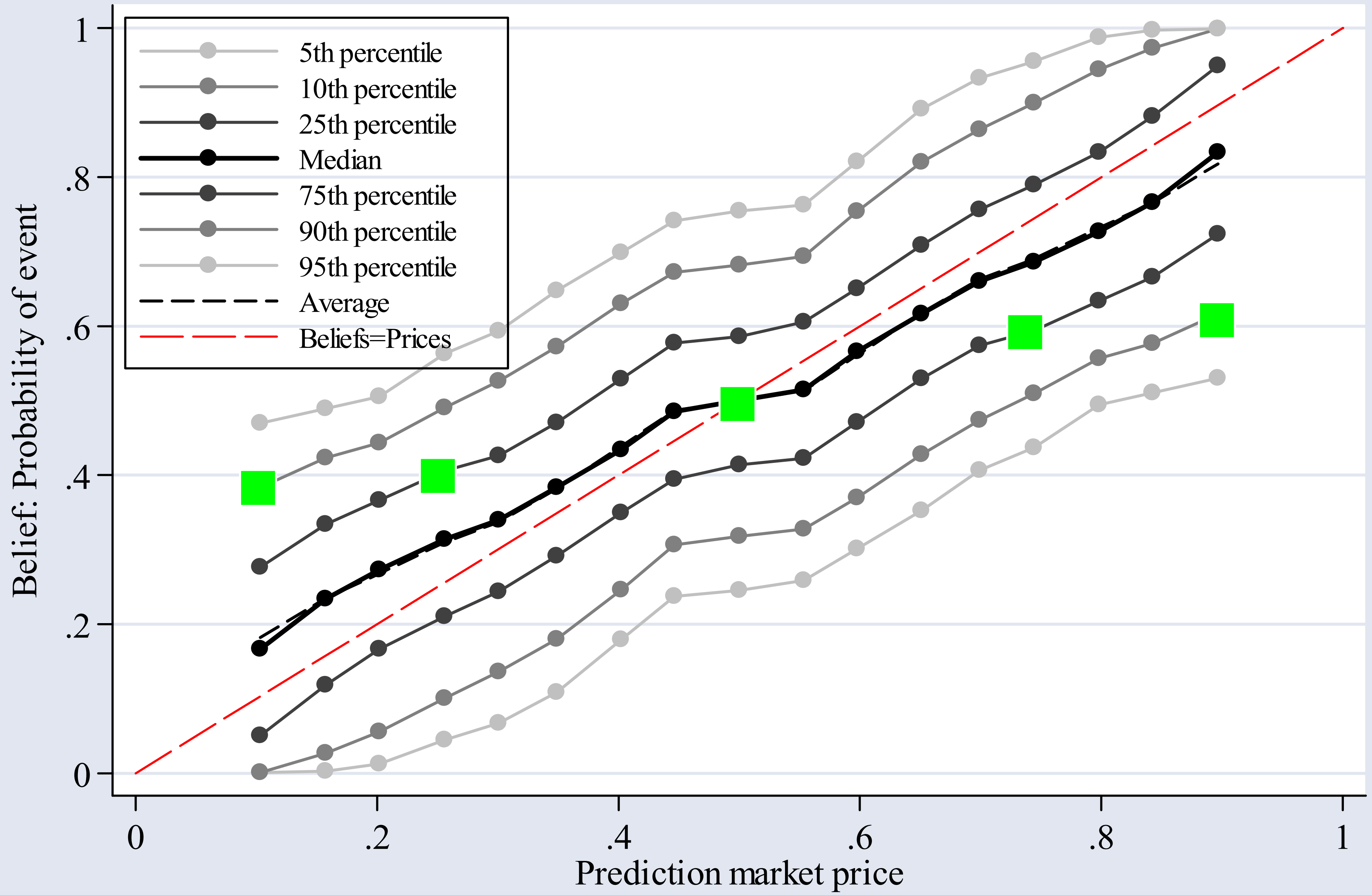
Distribution of Beliefs: NFL Football

- Probability Football Data
  - Online competition: Cash prizes for top 3 scores
  - Elicit belief about probability that team will win
  - Quadratic scoring rule: $100-400(w-q)^2$
- Data
  - 4 seasons (2000-2003)
  - Around 260 games per season
  - Average of 1320 players per game
  - Yields 1.4 million observations
  - We drop players who report 0 or 1 in >10% of games (sub-optimal strategy)

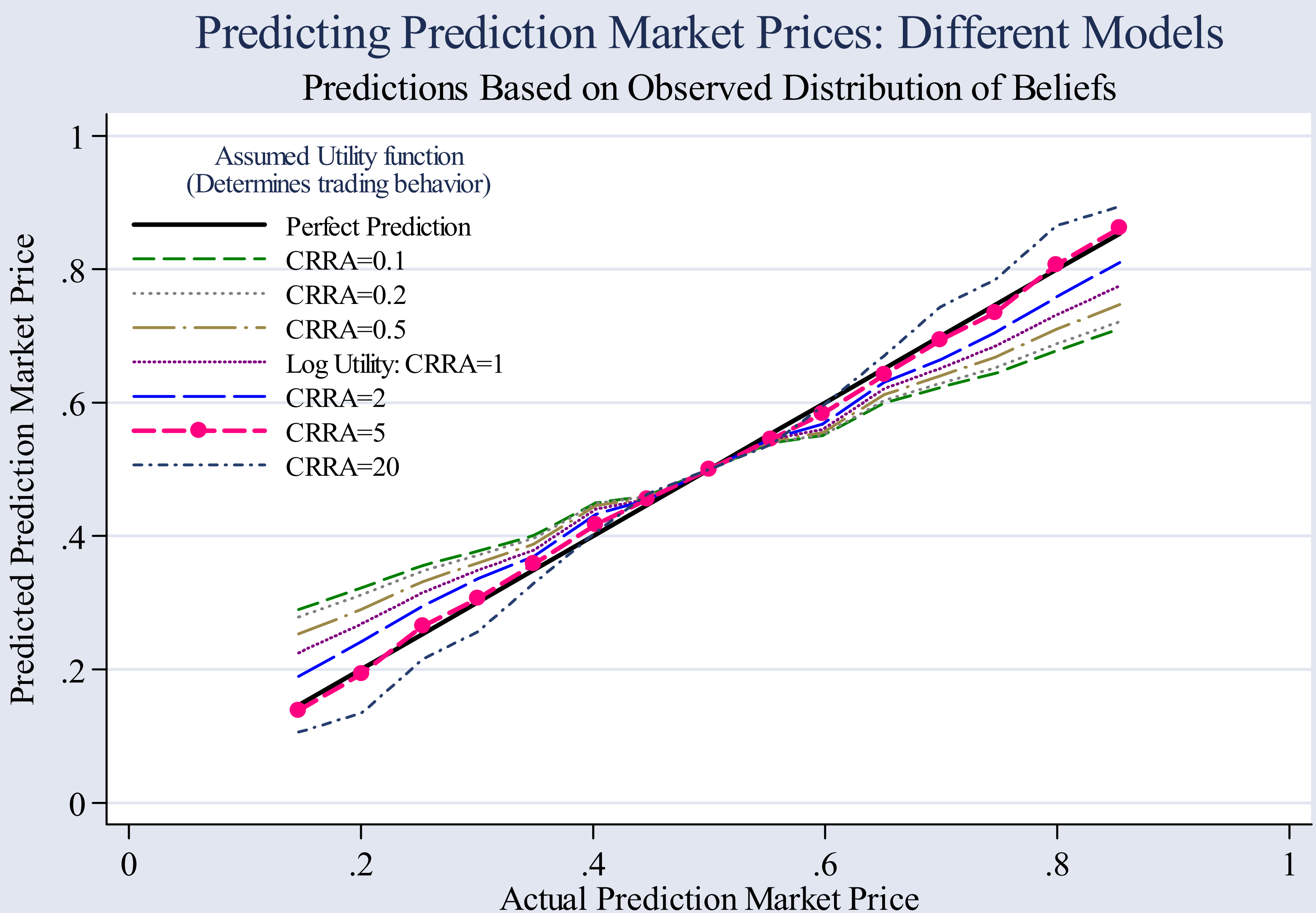
Distribution of Beliefs: NFL Football

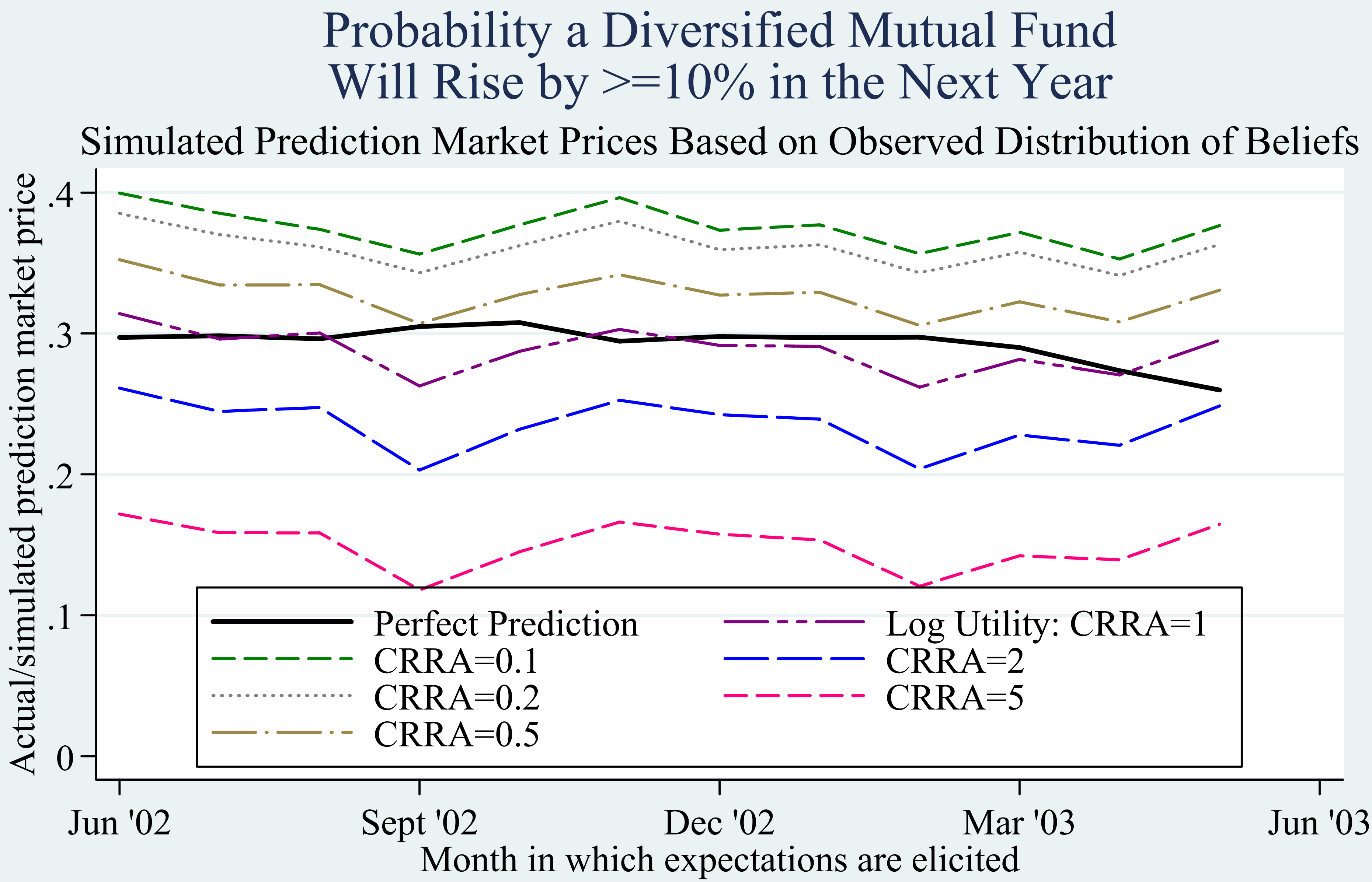


Distribution of Beliefs and Prediction Market Prices



Predicting Prices Based on Beliefs





Bold line shows the actual price of a synthetic security paying $1 if S&P rises by >=10% over the next year.
Price of this synthetic security is computed from S&P 500 options.
Dashed lines show predicted prices of this security, based on observed beliefs and assumptions about risk aversion.

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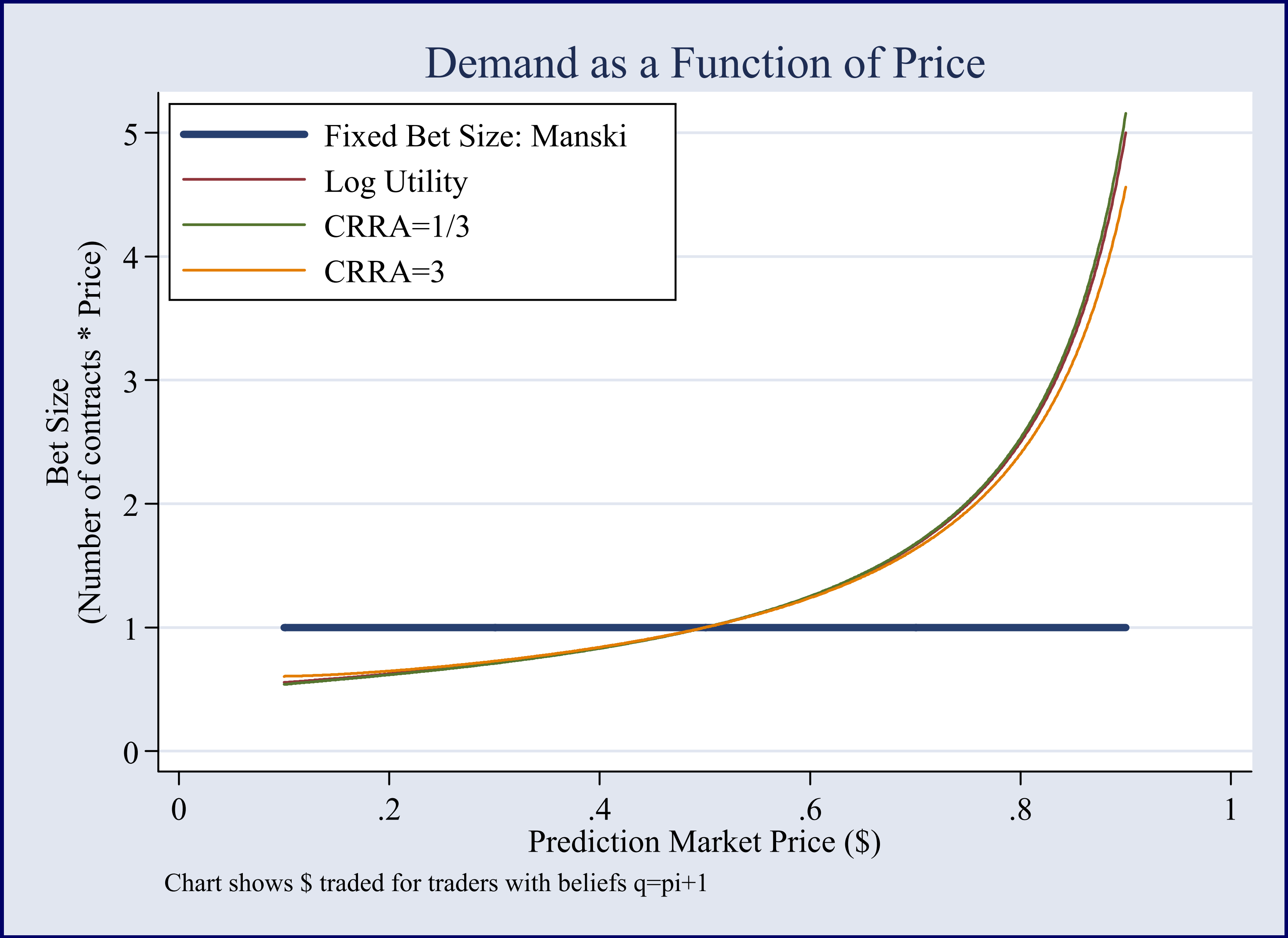
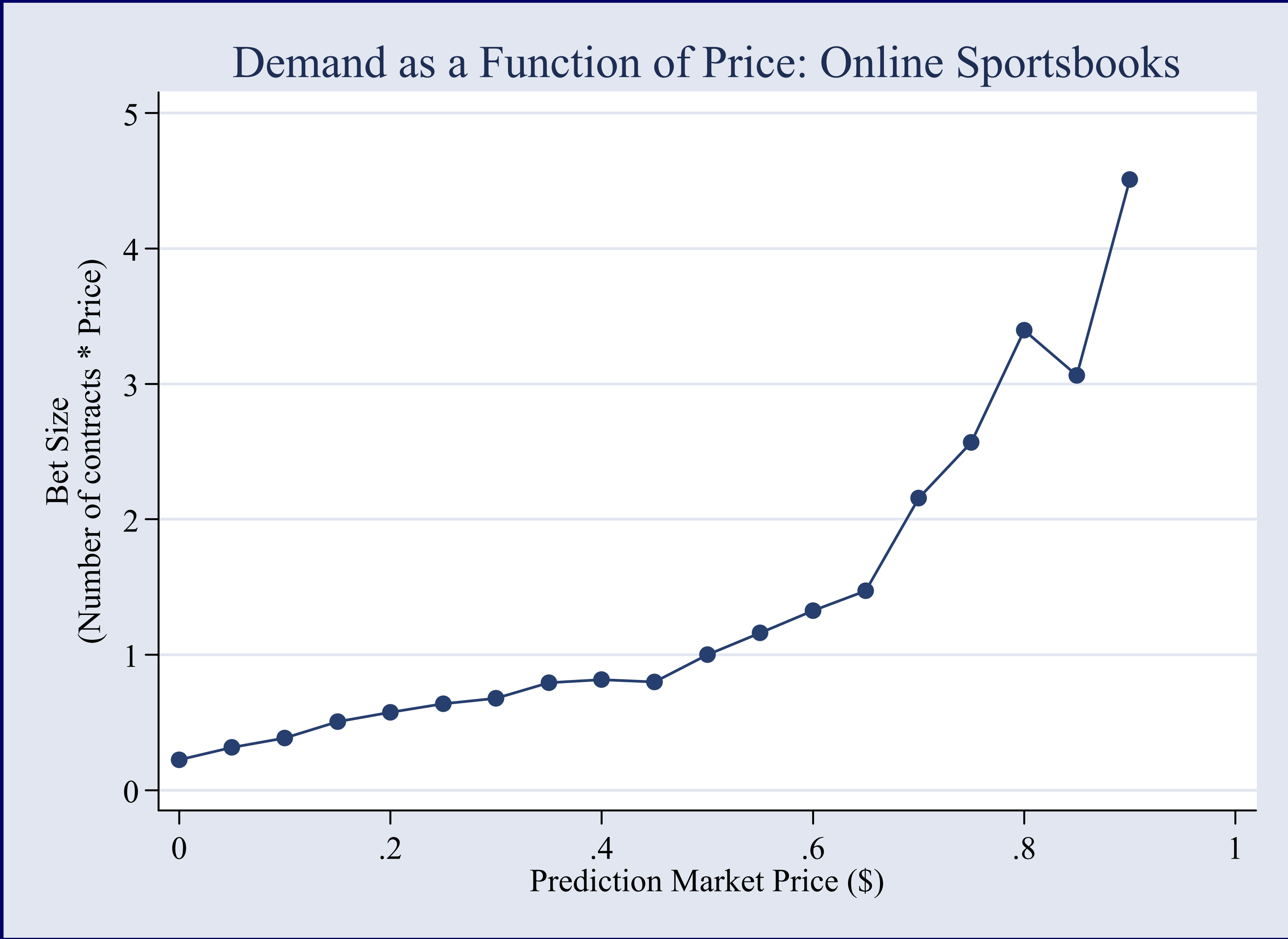
Utility Functions Determine Bet Size



Bookmaker Data

- Account-level sports betting data from 6 online sportsbooks
- $40 million gambled in 700,000 bets from 500 clients across many bet types
- Regression:
  - *Ln(BetSize) = Prediction market price*
    + individual gambler fixed effects

Relationship Between Bet Size and Price



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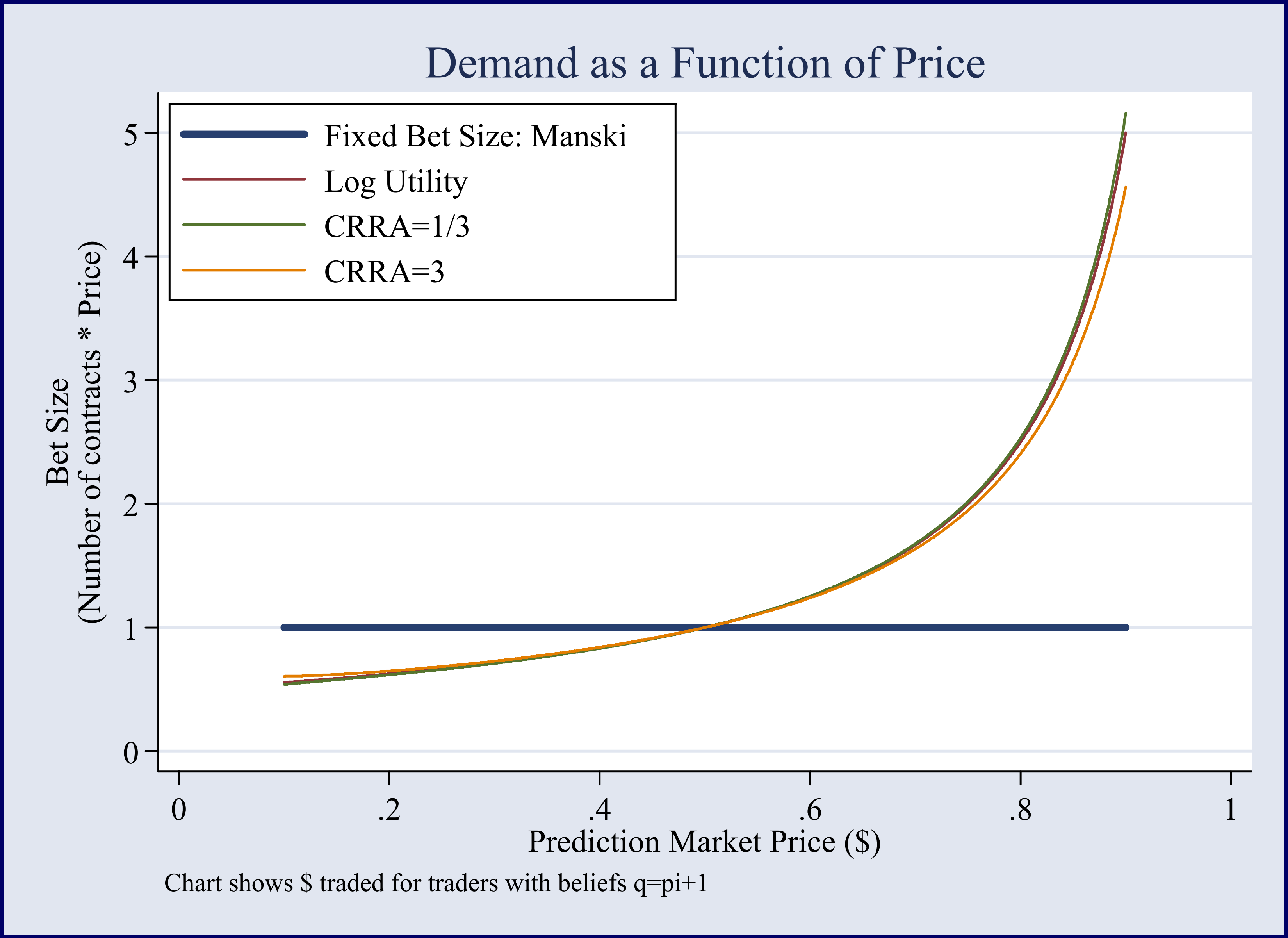

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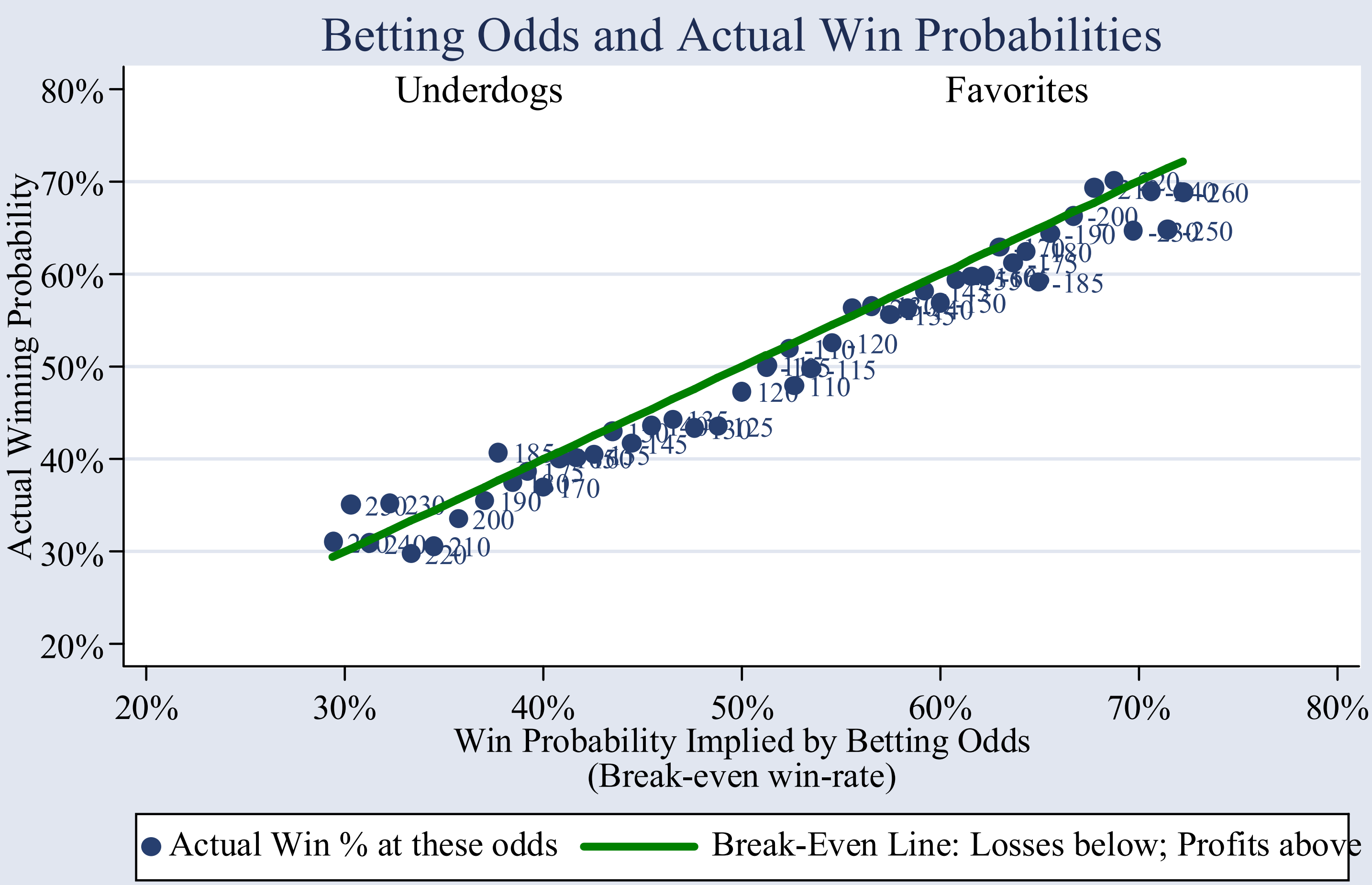
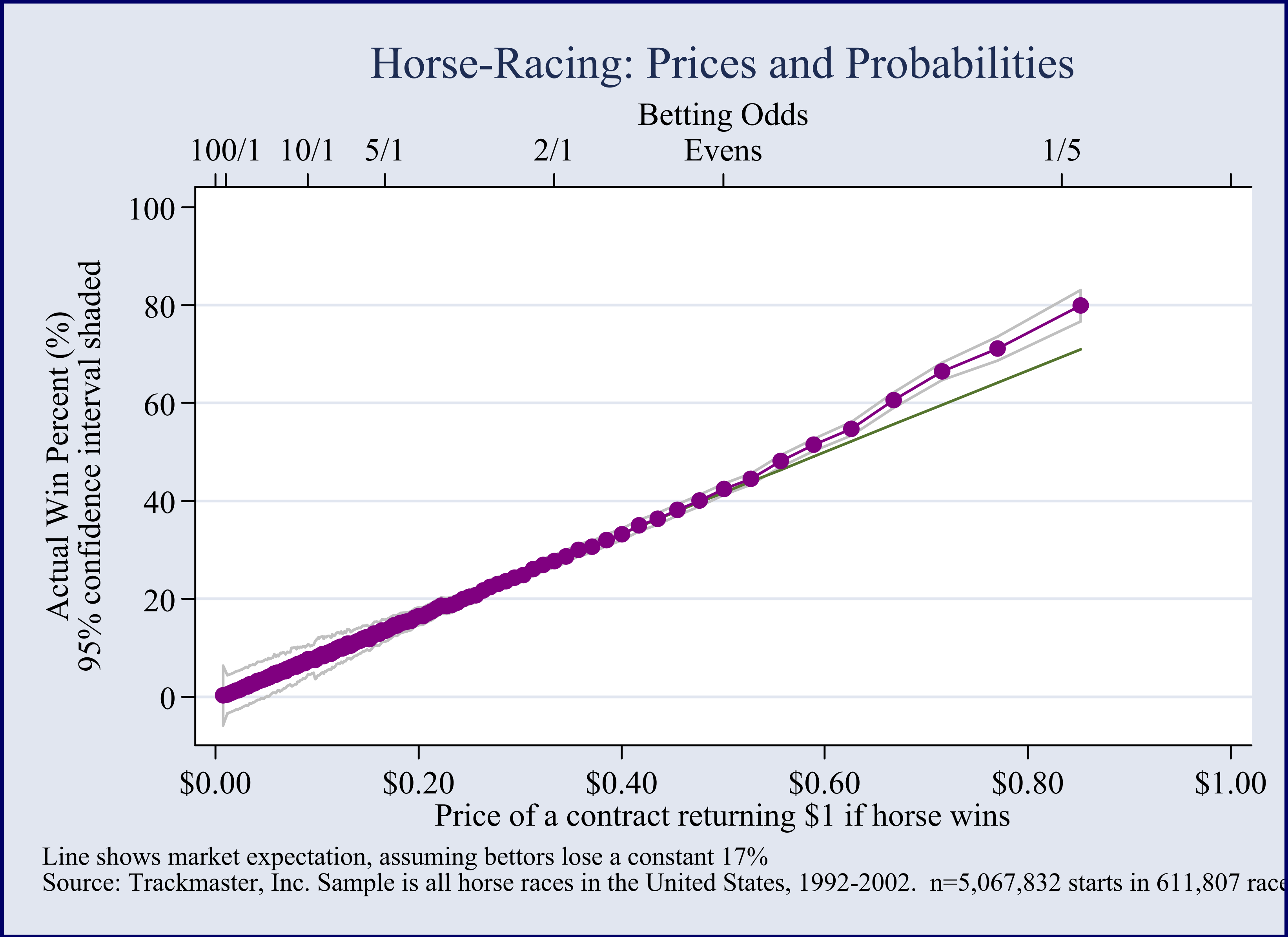
Prices and Probabilites: Baseball

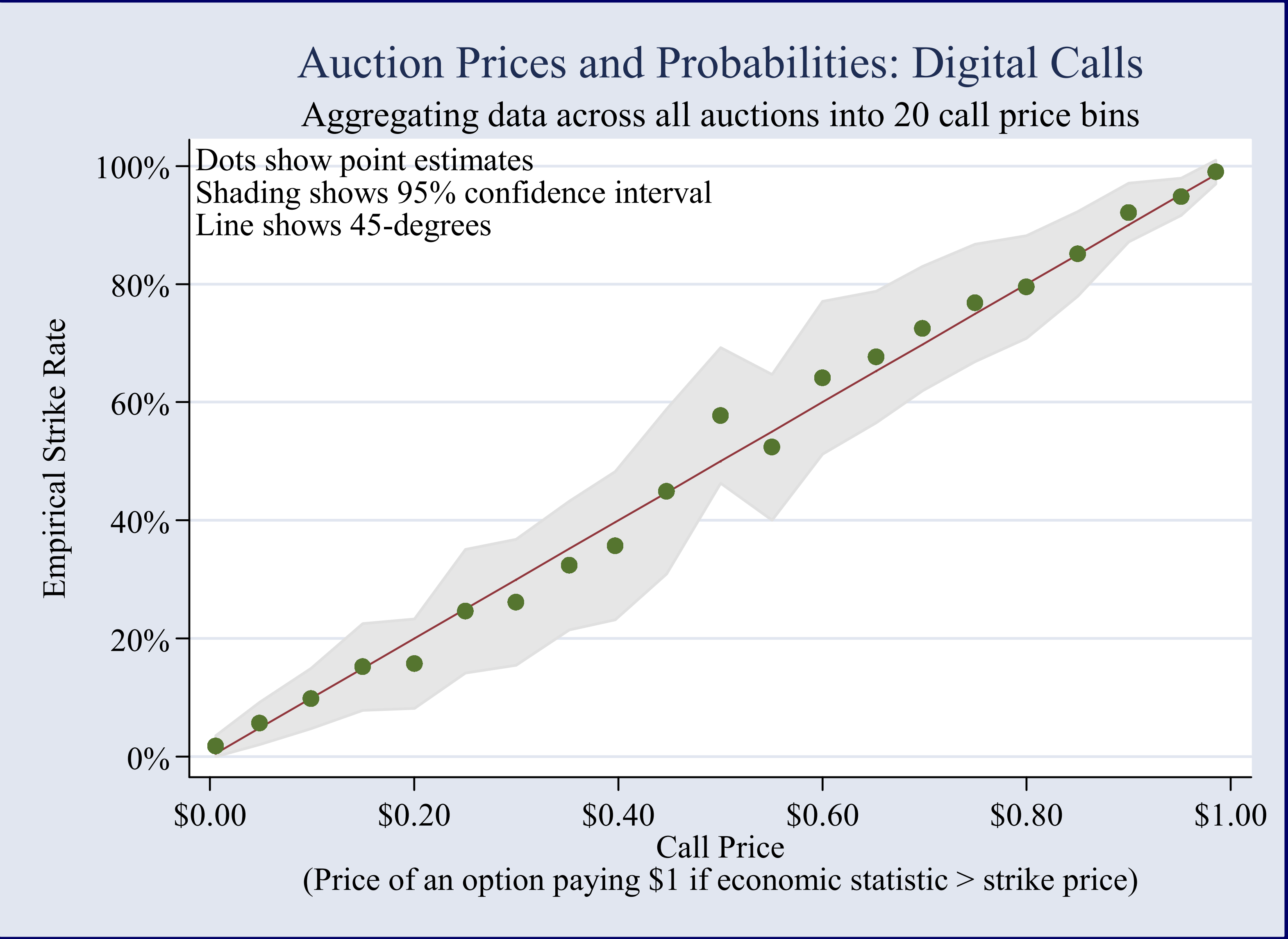


Results from 21885 MLB games played 1991-2000.

Prices and Probabilities: Horse Racing



Prices and Probabilities: Economic Derivatives



<u>Reference:</u> Refet Gurkaynak and Justin Wolfers (2005) "Economic Derivatives", *International Seminar on Macroeconomics.*

Conclusions

- Under what conditions to market prices aggregate all private info?
  - Grossman (1976): Stockmarket
    - » CARA utility and private signals ~N(μ,σ)
  - This paper: Prediction markets
    - » Log utility; any distribution of beliefs
- Manski:
  - Analytic results: Prices may fail to aggregate info
- This paper:
  - Calibration results: Prices are "close" to mean beliefs for plausible utility functions and distributions of beliefs (especially for $\pi \approx ½$)
  - Empirical results:
    - » Market prices are generally close to mean *beliefs*
    - » Market prices are generally close to actual *outcomes*
    - » Market prices are *fairly efficient* estimators of outcomes